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COMMUNICATIONS IN PARTIAL DIFFERENTIAL EQUATIONS
Vol. 28, Nos. 3 & 4, pp. 617–636, 2003

Conditions for the Local Boundedness of Solutions of the Navier–Stokes System in Three Dimensions

Mike O'Leary*

Department of Mathematics, Towson University,
Towson, Maryland, USA

ABSTRACT

We show that a weak solution of the Navier-Stokes system is locally bounded if there is some $\epsilon > 0$ so that either

$$\operatorname*{ess\,sup}_{(x,t)\in\Omega_T} \sup_{\rho>0} \frac{1}{\rho^{5/3+\epsilon}} \iint_{\Omega_T\cap B_\rho(x)\times(t-\rho^2,t)} |\mathbf{v}(\xi,\tau)|^{10/3}\, d\xi\, dt < \infty,$$

or

$$\operatorname*{ess\,sup}_{(x,t)\in\Omega_T} \sup_{\rho>0} \frac{1}{\rho^{1+\epsilon}} \iint_{\Omega_T\cap B_\rho(x)\times(t-\rho^2,t)} |\nabla\mathbf{v}(\xi,\tau)|^2\, d\xi\, dt < \infty.$$

*Correspondence: Mike O'Leary, Department of Mathematics, Towson University, Towson, MD/21252-0001, USA; E-mail: moleary@towson.edu.

DOI: 10.1081/PDE-120020490 0360-5302 (Print); 1532-4133 (Online)
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I. INTRODUCTION AND RESULTS

A fundamental question in mathematical physics is the local regularity of solutions of the Navier-Stokes system

$$\mathbf{v}_t - \triangle \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p = 0, \tag{1}$$
$$\operatorname{div} \mathbf{v} = 0. \tag{2}$$

In particular it is unknown if a solution $\mathbf{v}$ in a three dimensional domain Ω for which $\mathbf{v} \in L_\infty(0, T; L_2(\Omega)) \cap L_2(0, T, W_2^1(\Omega))$ is a priori bounded.

Our result is that if there is some $\epsilon > 0$ so that either

$$\operatorname*{ess\,sup}_{(x,t)\in\Omega_T} \sup_{\rho>0} \frac{1}{\rho^{5/3+\epsilon}} \iint_{\Omega_T \cap Q_\rho(x,t)} |\mathbf{v}(\xi, \tau)|^{10/3}\, d\xi\, dt < \infty, \tag{3}$$

or

$$\operatorname*{ess\,sup}_{(x,t)\in\Omega_T} \sup_{\rho>0} \frac{1}{\rho^{1+\epsilon}} \iint_{\Omega_T \cap Q_\rho(x,t)} |\nabla\mathbf{v}(\xi, \tau)|^{2}\, d\xi\, dt < \infty, \tag{4}$$

then $\mathbf{v} \in L_{\infty,\mathrm{loc}}(\Omega_T)$, where $Q_\rho(x,t) = B_\rho(x) \times (t-\rho^2, t)$ and $\Omega_T = \Omega \times (0, T)$.

More generally, we have the following.

Theorem 1. *Let $\mathbf{v} \in L_\infty(0, T; L_2(\Omega)) \cap L_2(0, T; W_2^1(\Omega))$ be a weak solution of the Navier–Stokes system*

$$\mathbf{v}_t - \triangle \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p = 0,$$
$$\operatorname{div} \mathbf{v} = 0.$$

in $\Omega_T = \Omega \times (0, T)$ where $\Omega \subseteq \mathbf{R}^3$. Suppose that either

(i) *There is some $2 < q \leq 5$ and some $\lambda > 5 - q$ that*

$$\operatorname*{ess\,sup}_{(x,t)\in\Omega_T} \sup_{\rho>0} \frac{1}{\rho^{\lambda}} \iint_{\Omega_T \cap Q_\rho(x,t)} |\mathbf{v}(\xi, \tau)|^{q}\, d\xi\, d\tau < \infty, \tag{5}$$

or

(ii) *There is some $10/7 < q \leq 5/2$ and $\lambda > 5 - 2q$ so that*

$$\operatorname*{ess\,sup}_{(x,t)\in\Omega_T} \sup_{\rho>0} \frac{1}{\rho^{\lambda}} \iint_{\Omega_T \cap Q_\rho(x,t)} |\nabla\mathbf{v}(\xi, \tau)|^{q}\, d\xi\, d\tau < \infty, \tag{6}$$

where $Q_\rho(x,t) \equiv B_\rho(x) \times (t-\rho^2, t)$. Then $\mathbf{v} \in L_{\infty,\mathrm{loc}}(\Omega_T)$ and $\mathbf{v}$ is C^∞ in the spatial variables.

To put this result in proper context, let us briefly review the state of the regularity theory for the Navier-Stokes system. The first main class of results we shall discuss show the boundedness of solutions that are sufficiently integrable. The main

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results are due to Ohyama (11), Serrin (16), and Takahashi (21). Representative of these results is the following (21, Theorem 3.1).

Theorem 2. *Let* $\mathbf{v} \in L_\infty(0, T; L_2(\Omega)) \cap L_2(0, T; W_2^1(\Omega))$ *be a weak solution of*

$$\mathbf{v}_t - \triangle \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p = 0$$
$$\operatorname{div} \mathbf{v} = 0$$

in $\Omega_T = \Omega \times (0, T)$, *where* $\Omega \subseteq \mathbf{R}^N$. *If either*

(i) $\mathbf{v} \in L_q(0, T; L_r(\Omega))$ *for* q *and* r *with* $2/q + N/r = 1$, $N < r \leq \infty$, *or*
(ii) $\operatorname{ess\,sup}_{0<t<T} \|\mathbf{v}(\cdot, t)\|_{L_N(\Omega)}$ *is sufficiently small.*

Then $\mathbf{v} \in L_{\infty,\mathrm{loc}}(\Omega_T)$ *and* $\mathbf{v}$ *is* C^∞ *in the spatial variables.*

If we require that $\mathbf{v}$ is a solution of the initial-boundary value problem, then more information can be obtained. Let $H(\Omega)$ be the completion of $\mathcal{D}(\Omega) = \{\mathbf{u} \in C_0^\infty(\Omega) : \operatorname{div} \mathbf{u} = 0\}$ in $L_2(\Omega)$. We have the following.

Theorem 3. *Let* $\mathbf{v} \in L_\infty(0, T; L_2(\Omega)) \cap L_2(0, T; W_2^1(\Omega))$ *be a weak solution of*

$$\mathbf{v}_t - \triangle \mathbf{v} + (\mathbf{v} \cdot \nabla \mathbf{v}) + \nabla p = 0$$
$$\operatorname{div} \mathbf{v} = 0$$
$$\mathbf{v}\big|_{\partial\Omega \times (0,T)} = 0$$
$$\mathbf{v}\big|_{t=0} = \mathbf{v}_o \in H(\Omega)$$

in $\Omega_T = \Omega \times (0, T)$, *where* $\Omega \subseteq \mathbf{R}^N$ *is smooth. If either*

(i) $\mathbf{v} \in L_q(0, T; L_r(\Omega))$ *for* q *and* r *with* $2/q + N/r = 1$, $N < r \leq \infty$, *or*
(ii) $\mathbf{v} \in C^0([0, T]; L_N(\Omega))$.

Then $\mathbf{v} \in C^\infty(\bar{\Omega} \times (0, T])$.

A proof of this result and the history of its development can be found in (7, Theorem 5.2, Remark 5.8). The limiting case $r = N$ has received considerable recent attention; we mention (1,2,9,10,17).

The addition of the boundary condition in Theorem 3 is significant. Indeed, Serrin (16) noted that if $\mathbf{v}(x, t) = a(t)\nabla\psi(x)$ for some harmonic function $\psi(x)$ and some integrable function $a(t)$, then $\mathbf{v}$ satisfies the Navier-Stokes system (1)–(2), but possesses no additional regularity in time beyond what is assumed for $a(t)$. Of course, $\mathbf{v}$ vanishes on $\partial\Omega \times (0, T)$ only if $\psi(x)$ and consequently $\mathbf{v}$ vanishes identically in all of Ω_T.

We also remark that there are results analogous to Theorem 3 where the integrability requirements on $\mathbf{v}$ are replaced by the requirements that either

(i*) $\nabla \mathbf{v} \in L_{q'}(0, T; L_{r'}(\Omega_T))$ for q' and r' with $2/q' + N/r' = 2$, $N < r' \leq \infty$, or
(ii*) $\nabla \mathbf{v} \in C^0([0, T]; L_{N/2}(\Omega))$.

See also (3) and (7, Remark 5.6).

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Another approach is the method of Scheffer (13–15) and Caffarelli et al. (4) which shows that the set of singular points is small in some sense. A point is called a regular point if the solution $\mathbf{v}$ is essentially bounded in a neighborhood of the point; the remaining points are called singular points. In particular, in (4) it was shown for suitable weak solutions that the one dimensional Hausdorff measure of the singular set is zero. This was done by proving the following.

Theorem 4. *There is a constant* $\delta > 0$ *so that if* $\mathbf{v}$ *is a suitable weak solution of the Navier-Stokes system and*

$$\limsup_{\rho\downarrow 0} \frac{1}{\rho}\iint_{Q^*_\rho(x,t)} |\nabla \mathbf{v}(\xi,\tau)|^2 \, d\xi \, d\tau < \delta \tag{7}$$

then (x, t) *is a regular point, where we set* $Q^*_\rho(x,t) = B_\rho(x) \times (t - (7/8)\rho^2, t + (1/8)\rho^2)$.

A suitable weak solution is a weak solution that satisfies some additional conditions, most significant of which is a generalized energy inequality. Unfortunately, it is not known if every weak solution of the Navier-Stokes system is a suitable weak solution. In (4), they were only able to construct a suitable weak solution on a bounded domain by assuming some additional regularity of the initial data. Moreover, because it is not known if solutions of the Navier-Stokes system are unique, it is not known if weak solutions constructed by other methods, e.g., Galerkin methods, are suitable weak solutions.

Our result is an extension of Theorem 2 that is inspired by Theorem 4. However, it is not sufficiently strong to obtain an estimate of the Hausdorff dimension of the singular set because our assumptions are essentially global in nature, while Eq. (7) is local.

We also remark that hypotheses (3)–(6) can be considered to be requirements that $\mathbf{v}$ or $\nabla\mathbf{v}$ be a member of a "parabolic" Morrey space.

II. SKETCH OF PROOF

There are two basic elements of the proof. First is a local representation theorem which enables us to estimate the solution at a point in terms of its integral average in a parabolic cylinder with vertex at that point, and a singular integral. In particular, denoting the integral average by

$$⨍\!\!\!\!⨍_{\mathcal{U}} f \, dx \, dt \equiv \frac{1}{\operatorname{meas}\mathcal{U}} \iint_{\mathcal{U}} f \, dx \, dt$$

we have the following result.

Proposition 5. *Let* $\mathbf{v} \in L_\infty(0, T; L_2(\Omega)) \cap L_2(0, T; W^1_2(\Omega))$ *be a weak solution of the Navier–Stokes system*

$$\mathbf{v}_t - \triangle\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = 0,$$
$$\operatorname{div}\mathbf{v} = 0,$$

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in $\Omega_T = \Omega \times (0, T)$, *where* $\Omega \subseteq \mathbf{R}^3$. *We have an absolute constant* γ *so that for almost every* $(x, t) \in \Omega_T$ *with* $Q_R(x, t) \subset\subset \Omega_T$,

$$|\mathbf{v}(x,t)| \le \gamma \fint\!\!\fint_{Q_R \setminus Q_{R/2}(x,t)} |\mathbf{v}(\xi,\tau)|\, d\xi\, d\tau + \gamma \iint_{Q_R(x,t)} \frac{|\mathbf{v}(\xi,\tau)|^2}{(|x-\xi| + \sqrt{t-\tau})^4}\, d\xi\, d\tau \quad (8)$$

and

$$|\mathbf{v}(x,t)| \le \gamma \fint\!\!\fint_{Q_R \setminus Q_{R/2}(x,t)} |\mathbf{v}(\xi,\tau)|\, d\xi\, d\tau + \gamma \iint_{Q_R(x,t)} \frac{|[(\mathbf{v}\cdot\nabla)\mathbf{v}](\xi,\tau)|}{(|x-\xi| + \sqrt{t-\tau})^3}\, d\xi\, d\tau. \quad (9)$$

To prove this result, we use the fact that the fundamental solution of the Stokes system can be written in the form curl **A** for a vector potential **A** which we can calculate explicitly. We then choose a cutoff function ζ and use curl(ζ**A**) as a test function. The details of the proof are provided in Sec. IV.

The second elements of the proof are the following propositions on fractional integration. These allow us to estimate the singular integrals that arise in the representation theorem and are generalizations of the usual Hardy–Littlewood–Sobolev results found, for example, in (20, Ch. 5, §1.2), and are inspired by the results in (12).

Proposition 6. *Let* $\mathcal{V} \subset \mathbb{R}^N \times \mathbb{R}$ *be a bounded domain, and suppose that*

(i) $\displaystyle \operatorname*{ess\,sup}_{(x,t)\in\mathcal{V}} \sup_{\rho>0} \frac{1}{\rho^\lambda} \iint_{\mathcal{V}\cap Q_\rho(x,t)} |f(\xi,\tau)|^q\, d\xi\, d\tau \equiv |f|^q_{\mathfrak{L}^\lambda_q(\mathcal{V})} < \infty$, *and*

(ii) $\displaystyle \iint_{\mathcal{V}} |f(\xi,\tau)|^m\, d\xi\, d\tau < \infty$

for some $m \ge q > 1$ *and* $0 \le \lambda < N+2$. *For* $(x,t) \in \mathcal{V}$, *define*

$$Tf(x,t) = \iint_{\mathcal{V}} \frac{f(\xi,\tau)}{(|x-\xi| + \sqrt{|t-\tau|})^{N+2-\alpha}}\, d\xi\, d\tau.$$

Then for any $m < p < \infty$ *satisfying*

$$\frac{1}{p} > \frac{q}{m}\left(\frac{1}{q} - \frac{\alpha}{N+2-\lambda}\right) \quad (10)$$

there is a constant $\gamma = \gamma(N, p, q, m, \alpha, \lambda, \mathcal{V})$ *so that*

$$\|Tf\|_{L_p(\mathcal{V})} \le \gamma \|f\|^{m/p}_{L_m(\mathcal{V})} |f|^{1-(m/p)}_{\mathfrak{L}^\lambda_q(\mathcal{V})}. \quad (11)$$

Proposition 7. *Let* $\mathcal{V} \subset \mathbb{R}^N \times \mathbb{R}$ *be a bounded domain, and suppose that*

(i) $\displaystyle \operatorname*{ess\,sup}_{(x,t)\in\mathcal{V}} \sup_{\rho>0} \frac{1}{\rho^\lambda} \iint_{\mathcal{V}\cap Q_\rho(x,t)} |g(\xi,\tau)|^q\, d\xi\, d\tau \equiv |g|^q_{\mathfrak{L}^\lambda_q(\mathcal{V})} < \infty$, *and*

(ii) $\displaystyle \iint_{\mathcal{V}} |f(\xi,\tau)|^m\, d\xi\, d\tau < \infty$

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for some m and q with $1/m + 1/q < 1$ and some $0 \le \lambda < N+2$. For $(x,t) \in \mathcal{V}$, define

$$T(f,g)(x,t) = \iint_{\mathcal{V}} \frac{f(\xi,\tau)g(\xi,\tau)}{(|x-\xi| + \sqrt{|t-\tau|})^{N+2-\alpha}}\, d\xi\, d\tau.$$

Then for any $m < p < \infty$ satisfying

$$\frac{1}{p} > \frac{1}{m} + \frac{1}{q} - \frac{\alpha + \lambda/q}{N+2} \tag{12}$$

there is a constant $\gamma = \gamma(N,p,q,m,\alpha,\lambda,\mathcal{V})$ so that

$$\|T(f,g)\|_{L_p(\mathcal{V})} \le \gamma \|f\|_{L_m(\mathcal{V})} |g|_{\mathfrak{L}_q^\lambda(\mathcal{V})}. \tag{13}$$

These propositions are proven by splitting the region of integration into an infinite sequence of concentric shells constructed from parabolic cylinders. Within each shell, the resulting singular integral can be directly estimated, and the parameters are chosen to ensure the convergence of the resulting infinite sum. The details are provided in Sec. V.

To prove the main result, we apply Proposition 6 or 7 to the singular integrals that arise in the representation theorem. This shows that if $\mathbf{v} \in L_{m,\mathrm{loc}}(\Omega_T)$ then $\mathbf{v} \in L_{p,\mathrm{loc}}(\Omega_T)$ for some $p > m$. We iterate this process until we have sufficient local integrability to apply the result of Theorem 2.

A number of standard results on various potentials are required; for the convenience of the reader, these are collected in an Appendix.

III. PROOF OF THE MAIN RESULT

We begin by assuming that hypothesis (i) is satisfied. Suppose that $\mathbf{v} \in L_{m,\mathrm{loc}}(\Omega_T)$ for some $m \ge q$; we claim that there is a constant $\kappa > 1$ depending only on q and λ so that $\mathbf{v} \in L_{p,\mathrm{loc}}(\Omega_T)$ for all $p < \kappa m$. Indeed, let $\mathcal{U} \subset\subset \mathcal{V} \subset\subset \Omega_T$ for some subdomains $\mathcal{U}$ and $\mathcal{V}$. Choose $R = R(\mathcal{U},\mathcal{V})$ so that $Q_R(x,t) \subseteq \mathcal{V}$ for all $(x,t) \in \mathcal{U}$. Apply Theorem 5 to conclude for almost every $(x,t) \in \mathcal{U}$ that

$$|\mathbf{v}(x,t)| \le \gamma \fint\!\!\fint_{Q_R(x,t)} |\mathbf{v}(\xi,\tau)|\, d\xi\, d\tau + \gamma \iint_{\mathcal{V}} \frac{|\mathbf{v}(\xi,\tau)|^2}{(|x-\xi| + \sqrt{|t-\tau|})^4}\, d\xi\, d\tau. \tag{14}$$

The first of these is bounded uniformly for almost every $(x,t) \in \mathcal{U}$. Apply Proposition 6 to $f = |\mathbf{v}|^2$; then because $|\mathbf{v}|^2 \in L_{m/2}$ and $||\mathbf{v}|^2|_{\mathfrak{L}_{q/2}^\lambda} < \infty$, we see that the second term is in $L_p(\mathcal{V})$ for any

$$p < \frac{m}{2 - q/(5-\lambda)}.$$

Combining these yields our claim.

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Because (i) implies that $\mathbf{v} \in L_{q,\mathrm{loc}}(\Omega_T)$, we can apply the claim repeatedly starting with $m = q$ until we can conclude $\mathbf{v} \in L_{m,\mathrm{loc}}(\Omega_T)$ for some $m > 5$; the theorem under hypothesis (i) then follows from the regularity result of Theorem 2.

The proof of the second part is similar. Indeed, suppose that $\mathbf{v} \in L_{m,\mathrm{loc}}(\Omega_T)$ with $1/m + 1/q < 1$; we claim that there is a constant $\kappa' > 0$ dependent only on q and λ so that $\mathbf{v} \in L_{p,\mathrm{loc}}(\Omega_T)$ for all $1/p > 1/m - \kappa'$. To see this, choose $\mathcal{U}$, $\mathcal{V}$, and R as before; then Theorem 5 implies for almost every $(x, t) \in \mathcal{U}$ that

$$|\mathbf{v}(x,t)| \le \gamma \fint\!\!\fint_{Q_R(x,t)} |\mathbf{v}(\xi,\tau)|\, d\xi\, d\tau + \gamma \iint_{\mathcal{V}} \frac{|\mathbf{v}(\xi,\tau)||\nabla\mathbf{v}(\xi,\tau)|}{(|x-\xi| + \sqrt{|t-\tau|})^3}\, d\xi\, d\tau. \tag{15}$$

Applying Proposition 7 to the second term with $f = |\mathbf{v}|$ and $g = |\nabla\mathbf{v}|$ for $\alpha = 2$, we see that the singular integral is in $L_p(\mathcal{V})$ for

$$\frac{1}{p} > \frac{1}{m} + \frac{1}{q} - \frac{2 + \lambda/q}{5} = \frac{1}{m} - \frac{1}{5q}(\lambda - (5 - 2q))$$

so that the claim follows.

Because $\mathbf{v} \in L_\infty(0,T; L_2(\Omega)) \cap L_2(0,T; W_2^1(\Omega)) \hookrightarrow L_{10/3}(\Omega_T)$, we can apply the claim repeatedly starting with $m = 10/3$ until we can use Theorem 2.

IV. PROOF OF THE REPRESENTATION THEOREM

Let $(x_o, t_o) \in \Omega_T$, and let $T_{jk}(x,t)$ be the fundamental solution of the Stokes system. Then

$$T_{jk}(x_o - x, t_o - t) = \delta_{jk}\Gamma(x_o - x, t_o - t) + \frac{1}{4\pi}\frac{\partial^2}{\partial x_j\,\partial x_k}\int_{\mathbb{R}^3} \frac{\Gamma(y, t_o - t)}{|(x_o - x) - y|}\, dy$$

where δ_{jk} is the Kronecker delta and

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{3/2}} \exp\left(-\frac{|x|^2}{4t}\right)$$

is the fundamental solution of the heat equation in $\mathbb{R}^3$.

For convenience, we set $\mathbf{T}_k = (T_{1k}, T_{2k}, T_{3k}) = (T_{k1}, T_{k2}, T_{k3})$. Because $\mathbf{T}_k$ is solenoidal, $\triangle\mathbf{T}_k = -\mathrm{curl}\,\mathrm{curl}\,\mathbf{T}_k$, and we can use the Newtonian potential to write $\mathbf{T}_k$ as

$$\mathbf{T}_k(x_o - x, t_o - t) = \frac{1}{4\pi}\mathrm{curl}\int_{\mathbb{R}^3} \frac{\mathrm{curl}\,\mathbf{T}_k(\xi, t_o - t)}{|(x_o - x) - \xi|}\, d\xi. \tag{16}$$

Details are provided in Appendix; see Lemmas 14 and 15.

Fix $Q_R(x_o,t_o) \subset\subset \Omega_T$, and let $0 \le \zeta(x,t) \le 1$ be a smooth cutoff function in $Q_R(x_o,t_o)$ so that $\zeta(x,t) = 1$ if $(x,t) \in Q_{R/2}(x_o,t_o)$, so that $\zeta(x,t) = 0$ if $(x,t) \notin Q_{3R/4}(x_o,t_o)$ and so that $|\zeta_t| + |\nabla\zeta|^2 + |D_x^2\zeta| \le C/R^2$ for some absolute constant C. Let $0 \le \zeta_n \le 1$ be a sequence of smooth approximations of ζ, with $\zeta_n \uparrow \zeta$, so that $\zeta_n(x,t) = 0$ if $|t_o - t| \le 1/2n$ and so that $\zeta_n(x,t) = \zeta(x,t)$ if $|t_o - t| \ge 1/n$. For $\eta > 0$, let

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J_η be a symmetric mollifying kernel in space and time, and denote the space–time mollification $J_\eta * f$ by f_η.

Define

$$\mathbf{\Phi}_k^{(n)}(x,t) = \frac{1}{4\pi}\operatorname{curl}\left\{\zeta_n(x,t)\int_{\mathbb{R}^3}\frac{\operatorname{curl}\mathbf{T}_k(\xi, t_o - t)}{|(x_o - x) - \xi|}\,d\xi\right\}. \tag{17}$$

Since $\mathbf{T}_k(x,t)$ is smooth away from $t = 0$, we know $\mathbf{\Phi}_k^{(n)} \in C_0^\infty(\Omega_T)$ and $\operatorname{div}\mathbf{\Phi}_k^{(n)} = 0$. Thus, if η is sufficiently small, then $(\mathbf{\Phi}_k^{(n)})_\eta$ is a valid test function, and

$$\iint_{\Omega_T}\mathbf{v}\cdot\left(\frac{\partial}{\partial t}+\triangle\right)\left(\mathbf{\Phi}_k^{(n)}\right)_\eta\,dx\,dt = \iint_{\Omega_T}[(\mathbf{v}\cdot\nabla)\mathbf{v}]\cdot\left(\mathbf{\Phi}_k^{(n)}\right)_\eta\,dx\,dt$$

or after a change of variables

$$\iint_{Q_R(x_o,t_o)}\mathbf{v}_\eta\cdot\left(\frac{\partial}{\partial t}+\triangle\right)\mathbf{\Phi}_k^{(n)}\,dx\,dt = \iint_{Q_R(x_o,t_o)}[(\mathbf{v}\cdot\nabla)\mathbf{v}]_\eta\cdot\mathbf{\Phi}_k^{(n)}\,dx\,dt. \tag{18}$$

We begin by estimating the left side of this equation.

First note that

$$\mathbf{\Phi}_k^{(n)}(x,t) = \zeta_n(x,t)\mathbf{T}_k(x_o - x, t_o - t) + \frac{1}{4\pi}\nabla\zeta_n(x,t)\int_{\mathbb{R}^3}\frac{\operatorname{curl}\mathbf{T}_k(\xi, t_o - t)}{|(x_o - x) - \xi|}\,d\xi.$$

Thus the linear portion of Eq. (18) can be written as

$$\begin{aligned}&\iint_{Q_R}\mathbf{v}_\eta\cdot\left(\frac{\partial}{\partial t}+\triangle\right)\mathbf{\Phi}_k^{(n)}\,dx\,dt\\ &\quad= \iint_{Q_R}\mathbf{v}_\eta\cdot\left(\frac{\partial}{\partial t}+\triangle\right)[\zeta_n(x,t)\mathbf{T}_k(x_o - x, t_o - t)]\,dx\,dt\\ &\qquad+\frac{1}{4\pi}\iint_{Q_R}\mathbf{v}_\eta\cdot\left(\frac{\partial}{\partial t}+\triangle\right)\left\{\nabla\zeta_n(x,t)\int_{\mathbb{R}^3}\frac{\operatorname{curl}\mathbf{T}_k(\xi, t_o - t)}{|(x_o - x) - \xi|}\,d\xi\right\}dx\,dt\\ &\quad= I + J.\end{aligned}$$

We shall estimate each of these terms separately.

Integrate by parts in I so that

$$I = -\iint_{Q_R}\left\{\left(\frac{\partial}{\partial t}-\triangle\right)\mathbf{v}_\eta\right\}\cdot\zeta_n(x,t)\mathbf{T}_k(x_o - x, t_o - t)\,dx\,dt.$$

We want to send $n\to\infty$; note that for each $(x,t)\in Q_R(x_o,t_o)$

$$\left|\left\{\left(\frac{\partial}{\partial t}-\triangle\right)\mathbf{v}_\eta\right\}\cdot\zeta_n(x,t)\mathbf{T}_k(x_o - x, t_o - t)\right| \le \frac{\gamma\|\mathbf{v}_\eta\|_{C^{2,1}_{x,t}}}{(|x_o - x| + \sqrt{t_o - t})^3}$$

because $|D_t^\ell D_x^m T_{jk}(x,t)| \le C(|x| + \sqrt{t})^{-3-m-2\ell}$ (Appendix, Lemma 12). Since the right side is integrable uniformly in n we can use Lebesgue's dominated convergence

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theorem to pass to the limit as $n \to \infty$ in I. Consequently when we insert the definition of $\mathbf{T}_k$ we obtain

$$\begin{aligned}\lim_{n\to\infty} I &= -\iint_{Q_R}\left\{\left(\frac{\partial}{\partial t}-\triangle\right)\mathbf{v}_\eta\right\}\cdot\zeta(x,t)\mathbf{T}_k(x_o-x,t_o-t)\,dx\,dt\\ &= -\iint_{Q_R}\left\{\left(\frac{\partial}{\partial t}-\triangle\right)(v_k)_\eta\right\}\zeta(x,t)\Gamma(x_o-x,t_o-t)\,dx\,dt\\ &\quad -\frac{1}{4\pi}\sum_{j=1}^{3}\iint_{Q_R}\left\{\left(\frac{\partial}{\partial t}-\triangle\right)(v_j)_\eta\right\}\\ &\quad\times\zeta(x,t)\frac{\partial^2}{\partial x_j\,\partial x_k}\int_{\mathbb{R}^3}\frac{\Gamma(y,t_o-t)}{|(x_o-x)-y|}\,dy\,dx\,dt\\ &= I_1+I_2.\end{aligned}$$

To estimate I_1, we rewrite it as

$$\begin{aligned}I_1 &= -\iint_{Q_R}\left\{\left(\frac{\partial}{\partial t}-\triangle\right)[(v_k)_\eta\zeta]\right\}\Gamma(x_o-x,t_o-t)\,dx\,dt\\ &\quad+\iint_{Q_R}(v_k)_\eta\left\{\left(\frac{\partial}{\partial t}-\triangle\right)\zeta\right\}\Gamma(x_o-x,t_o-t)\,dx\,dt\\ &\quad-2\sum_{j=1}^{3}\iint_{Q_R}\left\{\frac{\partial}{\partial x_j}(v_k)_\eta\right\}\frac{\partial\zeta}{\partial x_j}\Gamma(x_o-x,t_o-t)\,dx\,dt\\ &= I_1^{(1)}+I_1^{(2)}+I_1^{(3)}.\end{aligned}$$

Standard properties of the fundamental solution of the heat equation (Appendix, Lemma 13) imply that

$$I_1^{(1)} = -(v_k)_\eta(x_o,t_o)\zeta(x_o,t_o) = -(v_k)_\eta(x_o,t_o).$$

Because $|D_t^\ell D_x^m\Gamma(x,t)| \le C(|x|+\sqrt{t})^{-3-m-2\ell}$ (Appendix, Lemma 10), we know that

$$\left|I_1^{(2)}\right| \le \frac{\gamma}{R^2}\iint_{Q_R\setminus Q_{R/2}}\frac{|\mathbf{v}_\eta(x,t)|}{(|x_o-x|+\sqrt{t_o-t})^3}\,dx\,dt \le \gamma \fint\!\!\fint_{Q_R\setminus Q_{R/2}}|\mathbf{v}_\eta(x,t)|\,dx\,dt$$

because ζ is constant in $Q_{R/2}$.

To estimate $I_1^{(3)}$, we integrate by parts to obtain

$$\begin{aligned}I_1^{(3)} &= 2\iint_{Q_R}(v_k)_\eta(\triangle\zeta)\Gamma(x_o-x,t_o-t)\,dx\,dt\\ &\quad-2\sum_{j=1}^{3}\iint_{Q_R}(v_k)_\eta\frac{\partial\zeta}{\partial x_j}\frac{\partial\Gamma}{\partial x_j}(x_o-x,t_o-t)\,dx\,dt,\end{aligned}$$

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and applying the same estimates we used for $I_1^{(2)}$, we find that

$$\left|I_1^{(3)}\right| \le \gamma \iint_{Q_R \setminus Q_{R/2}} |\mathbf{v}_\eta(x,t)|\, dx\, dt.$$

Next we turn to I_2. Integrate by parts, and use the fact that $\mathbf{v}$, and hence $\mathbf{v}_\eta$ and the derivatives of $\mathbf{v}_\eta$, are solenoidal to obtain

$$I_2 = \frac{1}{4\pi}\sum_{j=1}^{3}\iint_{Q_R}\left\{\left(\frac{\partial}{\partial t} - \triangle\right)(v_j)_\eta\right\}\frac{\partial \zeta}{\partial x_j}\frac{\partial}{\partial x_k}\int_{\mathbb{R}^3}\frac{\Gamma(y, t_o - t)}{|(x_o - x) - y|}\, dy\, dx\, dt.$$

Integrating by parts once more yields

$$\begin{aligned} I_2 = &-\frac{1}{4\pi}\sum_{j=1}^{3}\iint_{Q_R}(v_j)_\eta \frac{\partial \zeta}{\partial x_j}\left(\frac{\partial}{\partial t} + \triangle\right)\frac{\partial}{\partial x_k}\int_{\mathbb{R}^3}\frac{\Gamma(y, t_o - t)}{|(x_o - x) - y|}\, dy\, dx\, dt \\ &-\frac{1}{4\pi}\sum_{j=1}^{3}\iint_{Q_R}(v_j)_\eta\left\{\left(\frac{\partial}{\partial t} + \triangle\right)\frac{\partial \zeta}{\partial x_j}\right\} \\ &\times \frac{\partial}{\partial x_k}\int_{\mathbb{R}^3}\frac{\Gamma(y, t_o - t)}{|(x_o - x) - y|}\, dy\, dx\, dt - \frac{1}{2\pi}\sum_{j,\ell=1}^{3}\iint_{Q_R}(v_j)_\eta \frac{\partial^2 \zeta}{\partial x_j\, \partial x_\ell} \\ &\times \frac{\partial^2}{\partial x_k\, \partial x_\ell}\int_{\mathbb{R}^3}\frac{\Gamma(y, t_o - t)}{|(x_o - x) - y|}\, dy\, dx\, dt. \end{aligned}$$

Because ζ is constant in $Q_{R/2}(x_o, t_o)$ and because we have the estimate $|D_t^\ell D_x^m \int |x-y|^{-1}\Gamma(y,t)\, dy| \le C(|x| + \sqrt{t})^{-1-m-2\ell}$ (Appendix, Lemma 11) we see that

$$|I_2| \le \gamma \iint_{Q_R \setminus Q_{R/2}} |\mathbf{v}_\eta(x,t)|\, dx\, dt.$$

To estimate J, we first note that

$$T_{jk}(x,t) = \delta_{jk}\Gamma(x,t) + \frac{1}{4\pi}\frac{\partial^2}{\partial x_j\, \partial x_k}\int_{\mathbb{R}^3}\frac{\Gamma(y,t)}{|x-y|}\, dy$$

so that if $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the usual orthonormal basis of $\mathbb{R}^3$, then

$$\mathbf{T}_k(x,t) = \Gamma(x,t)\mathbf{e}_k + \frac{1}{4\pi}\operatorname{grad}\frac{\partial}{\partial x_k}\int_{\mathbb{R}^3}\frac{\Gamma(y,t)}{|x-y|}\, dy$$

and hence

$$\operatorname{curl}\mathbf{T}_k(x,t) = \nabla\Gamma(x,t) \times \mathbf{e}_k. \tag{19}$$

Thus we can write J as

$$J = \frac{-1}{4\pi}\iint_{Q_R}\left[\left(\frac{\partial}{\partial t} - \triangle\right)\mathbf{v}_\eta\right]\left\{\nabla\zeta_n\left[\nabla\int_{\mathbb{R}^3}\frac{\Gamma(\xi, t_o - t)}{|(x_o - x) - \xi|}\, d\xi\right] \times \mathbf{e}_k\right\} dx\, dt.$$

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The usual estimates of Newtonian potentials of the heat kernel imply that the integrand is uniformly integrable in n, so we can pass to the limit as $n \to \infty$. Integrate by parts once more to discover that

$$\lim_{n\to\infty} J = \frac{1}{4\pi}\iint_{Q_R} \mathbf{v}_\eta \cdot \left\{ \left[\left(\frac{\partial}{\partial t} + \triangle\right)\nabla\zeta\right]\left[\nabla \int_{\mathbb{R}^3} \frac{\Gamma(\xi, t_o - t)}{|(x_o - x) - \xi|}\, d\xi\right] \times \mathbf{e}_k \right\} dx\, dt$$

$$+ \frac{1}{4\pi}\iint_{Q_R} \mathbf{v}_\eta \cdot \left\{ \nabla\zeta\left(\frac{\partial}{\partial t} + \triangle\right)\left[\nabla \int_{\mathbb{R}^3} \frac{\Gamma(\xi, t_o - t)}{|(x_o - x) - \xi|}\, d\xi\right] \times \mathbf{e}_k \right\} dx\, dt$$

$$+ \frac{1}{2\pi}\sum_{\ell=1}^{3}\iint_{Q_R} \mathbf{v}_\eta \cdot \left\{ \left(\frac{\partial}{\partial x_\ell}\nabla\zeta\right)\frac{\partial}{\partial x_\ell}\left[\nabla \int_{\mathbb{R}^3} \frac{\Gamma(\xi, t_o - t)}{|(x_o - x) - \xi|}\, d\xi\right] \times \mathbf{e}_k \right\} dx\, dt.$$

Once again use the standard estimates of Newtonian potentials of the heat kernel to conclude

$$\left|\lim_{n\to\infty} J\right| \le \gamma \fint\!\!\fint_{Q_R \setminus Q_{R/2}} |\mathbf{v}_\eta|\, dx\, dt.$$

To estimate the nonlinear part of Eq. (18), we begin by noting that the definition of $\mathbf{\Phi}_k^{(n)}$ implies that

$$K = \iint_{Q_R} [(\mathbf{v}\cdot\nabla)\mathbf{v}]_\eta \cdot \mathbf{\Phi}_k^{(n)}\, dx\, dt$$

$$= \iint_{Q_R} [(\mathbf{v}\cdot\nabla)\mathbf{v}]_\eta \cdot \zeta_n(x,t)\, \mathbf{T}_k(x_o - x, t_o - t)\, dx\, dt$$

$$+ \frac{1}{4\pi}\iint_{Q_R} [(\mathbf{v}\cdot\nabla)\mathbf{v}]_\eta \cdot \left\{ \nabla\zeta_n(x,t) \int_{\mathbb{R}^3} \frac{\operatorname{curl} \mathbf{T}_k(\xi, t_o - t)}{|(x_o - x) - \xi|}\, d\xi \right\} dx\, dt.$$

Use Eq. (19) and pass to the limit as $n \to \infty$ to find that

$$\lim_{n\to\infty} K = \iint_{Q_R} [(\mathbf{v}\cdot\nabla)\mathbf{v}]_\eta \cdot \zeta(x,t)\, \mathbf{T}_k(x_o - x, t_o - t)\, dx\, dt$$

$$+ \frac{1}{4\pi}\iint_{Q_R} [(\mathbf{v}\cdot\nabla)\mathbf{v}]_\eta \left\{ \nabla\zeta\left[\nabla \int_{\mathbb{R}^3} \frac{\Gamma(\xi, t_o - t)}{|(x_o - x) - \xi|}\, d\xi\right] \times \mathbf{e}_k \right\} dx\, dt$$

$$= K_1 + K_2$$

Standard estimates of the fundamental solution of the Stokes system (Lemma 12) imply that.

$$|K_1| \le \gamma \iint_{Q_R} \frac{|[(\mathbf{v}\cdot\nabla)\mathbf{v}]_\eta|}{(|x_o - x| + \sqrt{t_o - t})^3}\, dx\, dt.$$

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Alternatively, we can integrate by parts to see that

$$
\begin{aligned}
|K_1| &\le \sum_{j,m=1}^{3} \left| \iint_{Q_R} (v_m v_j)_\eta \zeta(x,t) \frac{\partial}{\partial x_m} T_{jk}(x_o - x, t_o - t)\, dx\, dt \right| \\
&\quad + \left| \sum_{j,m=1}^{3} \iint_{Q_R} (v_m v_j)_\eta \frac{\partial \zeta}{\partial x_m} T_{jk}(x_o - x, t_o - t)\, dx\, dt \right| \\
&\le \gamma \sum_{j,m=1}^{3} \iint_{Q_R} \frac{(v_m v_j)_\eta}{(|x_o - x| + \sqrt{t_o - t})^4}\, dx\, dt.
\end{aligned}
$$

As for K_2, we estimate the potential integral in the usual fashion (Lemma 11) to see that

$$
\begin{aligned}
|K_2| &\le \frac{\gamma}{R} \iint_{Q_R \setminus Q_{R/2}} \frac{|[(\mathbf{v}\cdot\nabla)\mathbf{v}]_\eta|}{(|x_o - x| + \sqrt{t_o - t})^2}\, dx\ dt \\
&\le \gamma \iint_{Q_R} \frac{|[(\mathbf{v}\cdot\nabla)\mathbf{v}]_\eta|}{(|x_o - x| + \sqrt{t_o - t})^3}\, dx\, dt.
\end{aligned}
$$

Alternatively, we integrate by parts to see that

$$
\begin{aligned}
|K_2| &\le \frac{1}{4\pi} \sum_{j,m=1}^{3} \frac{\gamma}{R^2} \iint_{Q_R \setminus Q_{R/2}} \frac{|(v_m v_j)_\eta|}{(|x_o - x| + \sqrt{t_o - t})^2}\, dx\, dt \\
&\quad + \frac{1}{4\pi} \sum_{j,m=1}^{3} \frac{\gamma}{R} \iint_{Q_R \setminus Q_{R/2}} \frac{|(v_m v_j)_\eta|}{(|x_o - x| + \sqrt{t_o - t})^3}\, dx\, dt \\
&\le \gamma \sum_{j,m=1}^{3} \iint_{Q_R} \frac{(v_m v_j)_\eta}{(|x_o - x| + \sqrt{t_o - t})^4}\, dx\, dt.
\end{aligned}
$$

Put the estimates for I, J, and K together, to discover for each $\eta > 0$ that

$$
\begin{aligned}
|\mathbf{v}_\eta(x_o, t_o)| &\le \gamma ⨌_{Q_R \setminus Q_{R/2}(x_o, t_o)} |\mathbf{v}_\eta(x,t)|\, dx\, dt \\
&\quad + \gamma \sum_{i,j=1}^{3} \iint_{Q_R(x_o, t_o)} \frac{|(v_i v_j)_\eta|}{(|x_o - x| + \sqrt{t_o - t})^4}\, dx\, dt
\end{aligned} \tag{20}
$$

and

$$
\begin{aligned}
|\mathbf{v}_\eta(x_o, t_o)| &\le \gamma ⨌_{Q_R \setminus Q_{R/2}(x_o, t_o)} |\mathbf{v}_\eta(x,t)|\, dx\, dt \\
&\quad + \gamma \iint_{Q_R} \frac{|[(\mathbf{v}\cdot\nabla)\mathbf{v}]_\eta|}{(|x_o - x| + \sqrt{t_o - t})^3}\, dx\, dt.
\end{aligned} \tag{21}
$$

Our result then follows by passing to the limit as $\eta \downarrow 0$. Note that Propositions 6 and 7 with $\lambda = 0$ ensure that the singular integrals define functions in $L_q(\Omega_T)$ for some q, so the indicated convergence will take place for almost every (x_o, t_o).

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V. FRACTIONAL INTEGRATION IN MORREY SPACES

In this section, we provide a proof of our results on fractional integration, Propositions 6 and 7.

A. Proof of Proposition 6

Let p be chosen to satisfy Eq. (10). Then

$$\begin{aligned}\|Tf\|_{L_p(\mathcal{V})} &= \sup_{\|h\|_{p'}=1} \iint_{\mathcal{V}} h(x,t)\, Tf(x,t)\, dx\, dt \\ &= \sup_{\|h\|_{p'}=1} \iint_{\mathcal{V}} \iint_{\mathcal{V}} \frac{h(x,t) f(\xi,\tau)}{(|x-\xi| + \sqrt{|t-\tau|})^{N+2-\alpha}}\, d\xi\, d\tau\, dx\, dt\end{aligned}$$

where $1/p + 1/p' = 1$. Choose h so that $\|h\|_{p'} = 1$. Then

$$\begin{aligned}&\iint_{\mathcal{V}} \iint_{\mathcal{V}} \frac{h(x,t) f(\xi,\tau)}{(|x-\xi| + \sqrt{|t-\tau|})^{N+2-\alpha}}\, d\xi\, d\tau\, dx\, dt \\ &\quad = \sum_{n=A_{\mathcal{V}}}^{\infty} \iint_{\mathcal{V}} \iint_{Q^n(x,t)} \frac{h(x,t) f(\xi,\tau)}{(|x-\xi| + \sqrt{|t-\tau|})^{N+2-\alpha}}\, d\xi\, d\tau\, dx\, dt \\ &\quad = \sum_{n=A_{\mathcal{V}}}^{\infty} I^{(n)}\end{aligned}$$

where

$$Q^n(x,t) = \left\{(\xi,\tau) \in \mathcal{V} : 2^{-n-1} \le |x-\xi| + \sqrt{|t-\tau|} \le 2^{-n}\right\} \tag{22}$$

and $A = A_{\mathcal{V}}$ is an integer chosen so that $B_{2^{-A}}(x) \times (t - 2^{-2A},\ t + 2^{-2A}) \supseteq \mathcal{V}$ for all $(x,t) \in \mathcal{V}$.

Rewrite the integrals $I^{(n)}$ as follows.

$$\begin{aligned}|I^{(n)}| &\le 2^{n(N+2-\alpha)} \iint_{\mathcal{V}} \iint_{Q^n(x,t)} |f(\xi,\tau)||h(x,t)|\, d\xi\, d\tau\, dx\, dt \\ &\le 2^{n(N+2-\alpha)} \iint_{\mathcal{V}} \iint_{Q^n(x,t)} \left[|f(\xi,\tau)|^{1-m/p} |h(x,t)|^{(p-m)/(p-1)q}\right] \\ &\qquad \times \left[|f(\xi,\tau)|^{m/p}\right]\left[|h(x,t)|^{(m-p-q+pq)/(p-1)q}\right] d\xi\, d\tau\, dx\, dt.\end{aligned}$$

Then because

$$\frac{p-m}{pq} + \frac{1}{p} + \frac{m-p-q+pq}{pq} = 1$$

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and because our hypotheses ensure that each term is positive, we can apply Hölder's inequality to discover that

$$
\begin{aligned}
|I^{(n)}| &\le 2^{n(N+2-\alpha)}\left(\iint_{\mathcal{V}}\iint_{Q^n(x,t)} |f(\xi,\tau)|^q |h(x,t)|^{p/(p-1)}\,d\xi\,d\tau\,dx\,dt\right)^{(p-m)/pq} \\
&\quad\times\left(\iint_{\mathcal{V}}\iint_{Q^n(x,t)} |f(\xi,\tau)|^m\,d\xi\,d\tau\,dx\,dt\right)^{1/p} \\
&\quad\times\left(\iint_{\mathcal{V}}\iint_{Q^n(x,t)} |h(x,t)|^{p/(p-1)}\,d\xi\,d\tau\,dx\,dt\right)^{(m-p-q+pq)/pq} \\
&= 2^{n(N+2-\alpha)} I_1^{(n)} I_2^{(n)} I_3^{(n)}.
\end{aligned}
$$

We shall estimate each of these in turn.

Write the first of these as

$$
\begin{aligned}
I_1^{(n)} &= \left(\iint_{\mathcal{V}}\iint_{Q^n(x,t)} |f(\xi,\tau)|^q |h(x,t)|^{p/(p-1)}\,d\xi\,d\tau\,dx\,dt\right)^{(p-m)/pq} \\
&\le \left(\operatorname*{ess\,sup}_{(x,t)\in\mathcal{V}} \iint_{Q^n(x,t)} |f(\xi,\tau)|^q\,d\xi\,d\tau\right)^{(p-m)/pq} \\
&\le \left(2\left(\frac{1}{2^n}\right)^{\lambda} |f|^q_{\mathfrak{L}_q^{\lambda}(\mathcal{V})}\right)^{(p-m)/pq} \\
&= \gamma(2^n)^{-\lambda((p-m)/pq)} |f|^{1-m/p}_{\mathfrak{L}_q^{\lambda}(\mathcal{V})}.
\end{aligned}
$$

On the other hand the region of integration for $I_2^{(n)}$ is the set

$$
\left\{(x,t,\xi,\tau) : (x,t),(\xi,\tau)\in\mathcal{V},\, 2^{-n-1} \le |x-\xi| + \sqrt{|t-\tau|} \le 2^{-n}\right\},
$$

so apply Fubini's theorem to interchange the order of integration and obtain

$$
I_2^{(n)} = \left(\iint_{\mathcal{V}}\iint_{Q^n(\xi,\tau)} |f(\xi,\tau)|^m\,dx\,dt\,d\xi\,d\tau\right)^{1/p} \le \gamma(2^n)^{-(N+2)/p} \|f\|^{m/p}_{L_m(\mathcal{V})}.
$$

Finally, note that

$$
\begin{aligned}
I_3^{(n)} &= \left(\iint_{\mathcal{V}}\iint_{Q^n(x,t)} |h(x,t)|^{p/(p-1)}\,d\xi\,d\tau\,dx\,dt\right)^{(m-p-q+pq)/pq} \\
&\le \gamma(2^n)^{-(N+2)((m-p-q+pq)/pq)}.
\end{aligned}
$$

Combine these results to find that

$$
|I^{(n)}| \le \gamma \|f\|^{m/p}_{L_m(\mathcal{V})} |f|^{1-m/p}_{\mathfrak{L}_q^{\lambda}(\mathcal{V})} 2^{nB}
$$

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where

$$B = (N+2-\alpha) - \lambda\frac{p-m}{pq} - (N+2)\left[\frac{1}{p} + \frac{m-p-q+pq}{pq}\right]$$
$$= (N+2-\lambda)\left[\left(\frac{1}{q} - \frac{\alpha}{N+2-\lambda}\right) - \frac{1}{p}\frac{m}{q}\right] < 0$$

by our requirements on p. As a consequence

$$\iint_{\mathcal{V}}\iint_{\mathcal{V}} \frac{h(x,t)f(\xi,\tau)}{(|x-\xi| + \sqrt{|t-\tau|})^{N+2-\alpha}}\, d\xi\, d\tau\, dx\, dt$$
$$\leq \gamma \|f\|_{L_m(\mathcal{V})}^{m/p} |f|_{\mathfrak{L}_q^\lambda(\mathcal{V})}^{1-m/p} \sum_{n=A_{\mathcal{V}}}^{\infty} 2^{nB} \leq \gamma \|f\|_{L_m(\mathcal{V})}^{m/p} |f|_{\mathfrak{L}_q^\lambda(\mathcal{V})}^{1-m/p}$$

and our proposition follows.

B. Proof of Proposition 7

We proceed as we did for the proof of Proposition 6. In particular, $\|T(f,g)\|_p \leq \sup_{\|h\|_{p'}=1} \sum_{n=A}^{\infty} I^{(n)}$ where

$$I^{(n)} = \iint_{\mathcal{V}}\iint_{Q^n(x,t)} \frac{h(x,t)f(\xi,\tau)g(\xi,\tau)}{(|x-\xi| + \sqrt{|t-\tau|})^{N+2-\alpha}}\, d\xi\, d\tau\, dx\, dt$$

and $Q^n(x,t)$ is defined by Eq. (22). Then

$$|I^{(n)}| \leq 2^{n(N+2-\alpha)} \iint_{\mathcal{V}}\iint_{Q^n(x,t)} \left[|f(\xi,\tau)|^{1-m/p} |h(x,t)|^{(p-m)/(p-1)m}\right]$$
$$\times \left[|g(\xi,\tau)||h(x,t)|^{p/(p-1)q}\right]\left[|f(\xi,\tau)|^{m/p}\right]$$
$$\times \left[|h(x,t)|^{(p/(p-1))(1-1/m-1/q)}\right] d\xi\, d\tau\, dx\, dt.$$

Because

$$\left(\frac{1}{m} - \frac{1}{p}\right) + \frac{1}{q} + \frac{1}{p} + \left(1 - \frac{1}{m} - \frac{1}{q}\right) = 1$$

and because our hypotheses ensure that each of these terms are positive, we can apply Hölder's inequality to conclude that

$$I^{(n)}| \leq 2^{n(N+2-\alpha)} \left(\iint_{\mathcal{V}}\iint_{Q^n(x,t)} |f(x,t)|^m |h(x,t)|^{p/(p-1)}\, d\xi\, d\tau\, dx\, dt\right)^{1/m-1/p}$$
$$\times \left(\iint_{\mathcal{V}}\iint_{Q^n(x,t)} |g(x,t)|^q |h(x,t)|^{1/q}\, d\xi\, d\tau\, dx\, dt\right)^{1/q}$$
$$\times \left(\iint_{\mathcal{V}}\iint_{Q^n(x,t)} |f(x,t)|^m\, d\xi\, d\tau\, dx\, dt\right)^{1/p}$$
$$\times \left(\iint_{\mathcal{V}}\iint_{Q^n(x,t)} |h(x,t)|^{p/(p-1)}\, d\xi\, d\tau\, dx\, dt\right)^{1-1/m-1/q}.$$

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Working as we did before, we then find that

$$|I^{(n)}| \leq 2^{n(N+2-\alpha)}\left(\|f\|_{L_m}^m\right)^{1/m-1/p}\left(2\frac{1}{2^{n\lambda}}|g|^q_{\mathfrak{L}_q^\lambda}\right)^{1/q}$$

$$\times\left(\gamma 2^{-n(N+2)}\|f\|_{L_m}^m\right)^{1/p}\left(\gamma 2^{-n(N+2)}\right)^{1-1/m-1/q}.$$

Thus

$$|I^{(n)}| \leq \gamma\|f\|_{L_m(\mathcal{V})}|g|_{\mathfrak{L}_q^\lambda(\mathcal{V})}2^{nB}$$

where

$$B = (N+2-\alpha) - \frac{\lambda}{q} - \frac{N+2}{p} - (N+2)\left(1 - \frac{1}{m} - \frac{1}{q}\right)$$

$$= (N+2)\left[-\frac{1}{p} + \frac{1}{m} + \frac{1}{q} - \frac{\alpha + \lambda/q}{N+2}\right].$$

Our restrictions on p then imply that $B < 0$, and the result follows.

APPENDIX

Here we collect precise statements of some standard results which are used in the article.

Because of the important role that they play in our work, we begin by recording some results for the Newtonian potential. These are commonly proven under the assumption that the functions involved have compact support; because that is not the case when these results are needed, we shall also provide brief sketches of the proofs.

Lemma 8. *Let $f \in C^k(\mathbb{R}^3)$, and define*

$$Tf(x) = \int_{\mathbb{R}^3}\frac{f(y)}{|x-y|}\,dy = \int_{\mathbb{R}^3}\frac{f(x-y)}{|y|}\,dy.$$

If, for all $0 \leq j \leq k$

$$\int_{\mathbb{R}^3}\frac{|D_x^j f(x)|}{|x|}\,dx < \infty$$

then $Tf \in C^k(\mathbb{R}^3)$ and $D_x^k(Tf) = T(D_x^k f)$.

Proof. It is sufficient to prove the result for $k = 1$ and for all $|x| \leq R$, for arbitrary R. If $|y| \leq R$ then

$$\left|\frac{f(x-y)}{|y|}\right| \leq \frac{\|f\|_{\infty, B_{2R}(0)}}{|y|}$$

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while if $|y| \geq R$ then

$$\left|\frac{f(x-y)}{|y|}\right| \leq \frac{|f(x-y)|}{|x-y|}\frac{|x-y|}{|y|} \leq \frac{|f(x-y)|}{|x-y|}\frac{|x|+|y|}{|y|} \leq 2\frac{|f(x-y)|}{|x-y|}.$$

Thus there is an integrable function $g_o(y)$ so that $|f(x-y)|/|y| \leq g_o(y)$ for all $|x| \leq R$. Applying the same process to f_{x_i}, we see that there are integrable functions $g_i(y)$ so that $|f_{x_i}(x-y)/|y|| \leq g_i(y)$ for all $|x| \leq R$. The usual rules for differentiating under the integral (8, Cap. XII, §9) give us our conclusion.

Lemma 9. *Let* $f \in C^2(\mathbb{R}^3)$, *and suppose that*

(i) $\lim_{|x|\to\infty} |f(x)| = 0$,

(ii) $\lim_{|x|\to\infty} |x|\,|\nabla f(x)| = 0$, *and*

(iii) $\displaystyle\int_{\mathbb{R}^3} \frac{|D^2 f(x)|}{|x|}\,dx < \infty$.

Then $f(x) = -(1/4\pi)\int_{\mathbb{R}^3} (\triangle f(y)/|x-y|)\,dy$.

Proof. Apply Stokes identity (5, Ch. 2, Sec. 2.1) to f on the ball $B_R(x)$ for some $R > 0$ to obtain

$$f(x) = \frac{1}{4\pi}\int_{\partial B_R(x)} \left\{\frac{\nabla f(y)\cdot\nu}{|x-y|} + \frac{f(y)}{|x-y|^2}\right\} d\sigma(y) - \frac{1}{4\pi}\int_{B_R} \frac{\triangle f(y)}{|x-y|}\,dy$$

where ν is the outward unit normal. Then pass to the limit as $R \to \infty$ to obtain the result.

We also need to understand the singularities of various potentials. We begin with the fundamental solution of the heat equation.

Lemma 10. *Let* Γ *be the fundamental solution of the heat equation in* $\mathbb{R}^3 \times \mathbb{R}$. *Then for any* $t > 0$, *for any* $0 \leq \sigma < 1/4$, *and for any integers* $\ell, m \geq 0$ *there is a constant* C *depending only on* ℓ, m, *and* σ *so that*

$$\left|D_t^{\ell} D_x^{m}\Gamma(x,t)\right| \leq \frac{Ce^{-\sigma|x|^2/t}}{(|x|+\sqrt{t})^{3+m+2\ell}}.$$

Proof. See (18, Ch. 2, Sec. 5).

Lemma 11. *Let* $T(x,t)$ *be given by*

$$T(x,t) = \int_{\mathbb{R}^3} \frac{\Gamma(y,t)}{|x-y|}\,dy$$

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where Γ is the fundamental solution of the heat equation in $\mathbb{R}^3 \times \mathbb{R}$. Then for any $t > 0$ and for any integers $\ell, m \geq 0$, there is a constant C depending only on ℓ and m so that

$$\left|D_t^\ell D_x^m T(x,t)\right| \leq \frac{C}{(|x| + \sqrt{t})^{1+m+2\ell}}.$$

Proof. See (19, Ch. 2, Sec. 5).

Lemma 12. *Let $T_{jk}(x,t)$ be the fundamental solution of the Stokes system in $\mathbb{R}^3 \times \mathbb{R}$. Then for any $t > 0$, for any integers $\ell, m \geq 0$ and for any spatial derivative of order m there exists a constant C depending only on ℓ and m so that*

$$\left|D_t^\ell D_x^m T_{jk}(x,t)\right| \leq \frac{C}{(|x| + \sqrt{t})^{3+m+2\ell}}.$$

Proof. This follows from the previous results and the form of T_{jk}. See also (19, Ch. 2, Sec. 5).

Lemma 13. *Let Γ be the fundamental solution of the heat equation in $\mathbb{R}^3 \times \mathbb{R}$, and suppose that $f \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R})$. Then for any $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$*

$$f(x,t) = \int_{-\infty}^{t} \int_{\mathbb{R}^3} \Gamma(x-\xi, t-\tau)\left(\frac{\partial}{\partial \tau} - \triangle_\xi\right) f(\xi,\tau)\, d\xi\, d\tau.$$

Proof. This follows immediately from the fact that Γ is the fundamental solution of the heat equation; in particular because $(\partial_t - \triangle)\Gamma = \delta$ as distributions. See also (6, Ch. 2, §2.3).

Lemma 14. *Let T_{jk} be the fundamental solution of the Stokes system in $\mathbb{R}^3$, and let $\mathbf{T}_k = (T_{1k}, T_{2k}, T_{3k})$. Then $\mathbf{T}_k(x,t)$ is solenoidal for all $t > 0$.*

Proof. By direct calculation, we see that

$$\begin{aligned}
\operatorname{div} \mathbf{T}_k(x,t) &= \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\left\{\delta_{jk}\Gamma(x,t) + \frac{1}{4\pi}\frac{\partial^2}{\partial x_j\, \partial x_k}\int_{\mathbb{R}^3} \frac{\Gamma(y,t)}{|x-y|}\, dy\right\} \\
&= \frac{\partial \Gamma}{\partial x_k}(x,t) + \frac{1}{4\pi}\frac{\partial}{\partial x_k}\triangle \int_{\mathbb{R}^3}\frac{\Gamma(y,t)}{|x-y|}\, dy.
\end{aligned}$$

After using the decay estimates for Γ to differentiate under the integral sign (Lemma 8), the representation of the Newtonian potential (Lemma 9) implies that

$$\operatorname{div} \mathbf{T}_k(x,t) = \frac{\partial \Gamma}{\partial x_k}(x,t) + \frac{\partial}{\partial x_k}(-\Gamma(x,t)) = 0$$

as required.

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Lemma 15. *Let $T_{jk}(x,t)$ be the fundamental solution of the Stokes system in $\mathbb{R}^3 \times \mathbb{R}$, and let $\mathbf{T}_k = (T_{1k}, T_{2k}, T_{3k})$. Then for any $t > 0$,*

$$\mathbf{T}_k(x,t) = \frac{1}{4\pi} \operatorname{curl} \int_{\mathbb{R}^3} \frac{\operatorname{curl} \mathbf{T}_k(\xi,t)}{|x-\xi|}\, d\xi.$$

Proof. Thanks to Lemma 9 and the decay estimates of Lemma 12,

$$\mathbf{T}_k(x,t) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\triangle \mathbf{T}_k(\xi,t)}{|x-\xi|}\, d\xi.$$

Use Lemma 14 to see that $\triangle \mathbf{T}_k = -\operatorname{curl}\operatorname{curl}\mathbf{T}_k + \operatorname{grad}\operatorname{div}\mathbf{T}_k = -\operatorname{curl}\operatorname{curl}\mathbf{T}_k$, then use the decay estimates to pull one of the derivatives outside the integral (Lemma 8) and obtain the result.

ACKNOWLEDGMENT

The author would like to thank Maria Schonbek for many helpful suggestions.

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Received December 2001
Revised June 2002

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